

# EMBEDDING MEASURE SPACES

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ABSTRACT. For a given measure space  $(X, \mathcal{B}, \mu)$  we construct all measure spaces  $(Y, \mathcal{C}, \lambda)$  in which  $(X, \mathcal{B}, \mu)$  is embeddable. The construction is modeled on the ultrafilter construction of the Stone–Čech compactification of a completely regular topological space. Under certain conditions the construction simplifies. Examples are given when this simplification occurs.

## 1. INTRODUCTION

A measurable space  $(X, \mathcal{B})$  is said to be *embedded* in a measurable space  $(Y, \mathcal{C})$  (denoted by  $(X, \mathcal{B}) \subseteq (Y, \mathcal{C})$ ) if  $X \subseteq Y$  and

$$\mathcal{B} = \{C \cap X : C \in \mathcal{C}\}.$$

A measure space  $(X, \mathcal{B}, \mu)$  is said to be *embedded* in a measure space  $(Y, \mathcal{C}, \lambda)$  (denoted by  $(X, \mathcal{B}, \mu) \subseteq (Y, \mathcal{C}, \lambda)$ ) if  $(X, \mathcal{B}) \subseteq (Y, \mathcal{C})$  and  $\mu(C \cap X) = \lambda(C)$  for each  $C \in \mathcal{C}$ . In this note, for a given measure space  $(X, \mathcal{B}, \mu)$ , we construct all measure spaces  $(Y, \mathcal{C}, \lambda)$  in which  $(X, \mathcal{B}, \mu)$  is embedded. Equivalently, for a given measure space  $(X, \mathcal{B}, \mu)$ , we construct all measure spaces  $(Y, \mathcal{C}, \lambda)$  which contains  $(X, \mathcal{B}, \mu)$  as a thick subspace. (Recall that a subset  $X$  of a measure space  $(Y, \mathcal{C}, \lambda)$  is said to be *thick* (or *of full outer measure*) if  $\lambda(C) = 0$  for each  $C \in \mathcal{C}$  such that  $C \subseteq Y \setminus X$ , equivalently, if  $\lambda_*(Y \setminus X) = 0$ , where  $\lambda_*$  denotes the inner measure induced by  $\lambda$ . If  $X$  is a thick subset of  $(Y, \mathcal{C}, \lambda)$  and if

$$\mathcal{B} = \{C \cap X : C \in \mathcal{C}\}$$

and  $\mu(C \cap X) = \lambda(C)$  for each  $C \in \mathcal{C}$ , then  $(X, \mathcal{B}, \mu)$  is a measure space which is embedded in  $(Y, \mathcal{C}, \lambda)$ . Conversely, if  $(X, \mathcal{B}, \mu)$  is embedded in  $(Y, \mathcal{C}, \lambda)$ , then  $X$  is a thick subset of  $(Y, \mathcal{C}, \lambda)$ ; see e.g. Theorem 17.A of [4].) Our construction here is analogous to the ultrafilter construction of the Stone–Čech compactification of a completely regular topological space  $X$ . (Completely regular topological spaces are always assumed to be Hausdorff.)

We recall some basic facts, definitions and notation. For details we refer the reader to [1], [3] and [4]. Let  $(X, \mathcal{B})$  be a measurable space. A non-empty  $\mathcal{A} \subseteq \mathcal{B}$  is called a *filter-base* in  $\mathcal{B}$  if for every  $A, B \in \mathcal{A}$  there exists a non-empty  $C \in \mathcal{A}$  such that  $C \subseteq A \cap B$ . A *filter*  $\mathcal{F}$  in  $\mathcal{B}$  is a filter-base such that  $B \in \mathcal{F}$  whenever  $B \in \mathcal{B}$  and  $F \subseteq B$  for some  $F \in \mathcal{F}$ . An *ultrafilter* in  $\mathcal{B}$  is a maximal (with respect to  $\subseteq$ ) filter. An ultrafilter is called *free* if it has empty intersection, otherwise, it is called *fixed*. An ultrafilter is said to have the *countable intersection property* (*c.i.p.*, in short) if every countable number of its elements has a non-empty intersection.

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It is known that every filter-base in  $\mathcal{B}$  is contained in some ultrafilter in  $\mathcal{B}$ , and that a filter-base  $\mathcal{A}$  in  $\mathcal{B}$  is an ultrafilter if and only if for each  $B \in \mathcal{B}$  if  $B$  meets every element of  $\mathcal{A}$  then  $B \in \mathcal{A}$ . Note that an ultrafilter  $\mathcal{F}$  in  $\mathcal{B}$  has c.i.p. if and only if it is  $\sigma$ -complete, i.e., it is closed under countable intersections.

Let  $X$  be a topological space. By a *zero-set* in  $X$  we mean a set of the form  $f^{-1}(0)$  where  $f : X \rightarrow [0, 1]$  is continuous; the complement of a zero-set is called a *cozero-set*; denote  $Z(f) = f^{-1}(0)$  and  $\text{Coz}(f) = X \setminus Z(f)$ . Let  $\mathcal{Z}(X)$  and  $\text{Coz}(X)$  denote the set of all zero-sets and the set of all cozero-sets of  $X$ , respectively. Let  $X$  be a completely regular topological space. A *compactification* of  $X$  is a compact Hausdorff topological space which contains  $X$  as a dense subspace. We denote by  $\beta X$  the *Stone–Čech compactification* of  $X$ , which always exists, and is characterized by either of the following properties:

- Every continuous function from  $X$  to a compact space is continuously extendible over  $\beta X$ .
- Every continuous function from  $X$  to  $[0, 1]$  is continuously extendible over  $\beta X$ .
- For every  $Z, S \in \mathcal{Z}(X)$  such that  $Z \cap S = \emptyset$  we have

$$\text{cl}_{\beta X} Z \cap \text{cl}_{\beta X} S = \emptyset.$$

- For every  $Z, S \in \mathcal{Z}(X)$  we have

$$\text{cl}_{\beta X}(Z \cap S) = \text{cl}_{\beta X} Z \cap \text{cl}_{\beta X} S.$$

Note, in particular, this implies that disjoint zero-sets (and thus disjoint closed-open subsets) in  $X$  have disjoint closures in  $\beta X$ . For a completely regular topological space  $X$  the *Hewitt realcompactification*  $vX$  of  $X$  is the subspace of  $\beta X$  defined by

$$vX = \bigcap \{C : C \in \text{Coz}(\beta X) \text{ and } X \subseteq C\}.$$

A topological space is said to be *realcompact* if it is homeomorphic to a closed subspace of some topological product of the real line. Every regular Lindelöf topological space is realcompact. It is known that a completely regular topological space  $X$  is realcompact if and only if  $X = vX$  if and only if for every  $p \in \beta X \setminus X$  there exists a zero-set  $Z$  in  $\beta X$  such that  $p \in Z$  and  $Z \cap X = \emptyset$ .

A *topological measurable space* is a triple  $(X, \mathcal{O}, \mathcal{B})$  where  $(X, \mathcal{B})$  is a measurable space and  $(X, \mathcal{O})$  is a topological space such that  $\mathcal{O} \subseteq \mathcal{B}$ , i.e., every open set (and thus every Borel set) is measurable.

This note is organized as follows. In Section 2 we construct all measure spaces  $(Y, \mathcal{C}, \lambda)$  in which a given measure space  $(X, \mathcal{B}, \mu)$  is embedded. In Section 3 we simplify the construction under certain additional conditions on  $(X, \mathcal{B}, \mu)$ . Indeed, we prove that if the points of  $X$  are separated by measurable sets in  $\mathcal{B}$  and there is no free ultrafilter in  $\mathcal{B}$  with c.i.p., then  $(X, \mathcal{B}, \mu)$  is embeddable in  $(Y, \mathcal{C}, \lambda)$  if and only if  $(Y, \mathcal{C}, \lambda)$  is obtained from  $(X, \mathcal{B}, \mu)$  by “blowing” certain points of  $X$  up and “pasting” a certain measurable space to  $X$  in a certain way. In Section 4 we provide examples satisfying the assumption of the theorem in Section 3, i.e., we find examples of measure spaces  $(X, \mathcal{B}, \mu)$  with no free ultrafilter in  $\mathcal{B}$  having c.i.p. It turns out that the class of such measure spaces  $(X, \mathcal{B}, \mu)$  is reasonably large (e.g., it contains the class of all first-countable realcompact topological measure spaces, thus in particular, containing all  $n$ -dimensional Lebesgue measure spaces) and behaves very nicely in connection with the standard operations on measure spaces (e.g., we show that for any  $\sigma$ -finite measure spaces  $(X, \mathcal{B}, \mu)$  and  $(Y, \mathcal{C}, \lambda)$

such that in each of which singletons are measurable, considering the measure space  $(X \times Y, \mathcal{B} \times \mathcal{C}, \mu \times \lambda)$ , there is no free ultrafilter in  $\mathcal{B} \times \mathcal{C}$  with c.i.p. if and only if there is a free ultrafilter with c.i.p. neither in  $\mathcal{B}$  nor in  $\mathcal{C}$ .) Finally, in Section 5 we give examples of measure spaces  $(X, \mathcal{B}, \mu)$  having arbitrarily large number of free ultrafilter in  $\mathcal{B}$  with c.i.p. We leave some problems open which are formally stated.

## 2. THE CONSTRUCTION OF MEASURE SPACES IN WHICH A GIVEN MEASURE SPACE $(X, \mathcal{B}, \mu)$ IS EMBEDDABLE

The following lemma is well known.

**Lemma 2.1.** *Let  $(X, \mathcal{B})$  be a measurable space. Let  $\mathcal{U}$  be an ultrafilter in  $\mathcal{B}$ .*

- (1) *For any  $B \in \mathcal{B}$  either  $B \in \mathcal{U}$  or  $X \setminus B \in \mathcal{U}$ .*
- (2) *Suppose that  $\mathcal{U}$  has c.i.p. and  $B_1, B_2, \dots \in \mathcal{B}$ . Then*

$$(2.1) \quad \bigcup_{n=1}^{\infty} B_n \in \mathcal{U}$$

*if and only if  $B_n \in \mathcal{U}$  for some  $n \in \mathbb{N}$ .*

*Proof.* To show (1), note that if  $B \notin \mathcal{U}$  for some  $B \in \mathcal{B}$ , then, since  $\mathcal{U}$  is an ultrafilter, we have  $B \cap U = \emptyset$  for some  $U \in \mathcal{U}$ . Thus  $U \subseteq X \setminus B$ , which implies that  $X \setminus B \in \mathcal{U}$ .

To show (2), note that (2.1) holds trivially if  $B_n \in \mathcal{U}$  for some  $n \in \mathbb{N}$ . To show the converse, suppose that (2.1) holds, while  $B_n \notin \mathcal{U}$  for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  (since  $\mathcal{U}$  is an ultrafilter) there exists some  $U_n \in \mathcal{U}$  such that  $B_n \cap U_n = \emptyset$ . Now

$$\bigcap_{i=1}^{\infty} U_i \cap \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left( \bigcap_{i=1}^{\infty} U_i \cap B_n \right) \subseteq \bigcup_{n=1}^{\infty} (U_n \cap B_n) = \emptyset$$

contradicting the fact that  $\mathcal{U}$  has c.i.p. □

**Theorem 2.2.** *Let  $(X, \mathcal{B}, \mu)$  be a measure space. Then  $(Y, \mathcal{C}, \lambda)$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded if and only if there exists a measurable space  $(Z, \mathcal{D})$ , a collection  $\{\mathcal{D}_B : B \in \mathcal{B}\}$  of non-empty subsets of  $\mathcal{D}$  such that*

- (1)  $\emptyset \in \mathcal{D}_\emptyset$ ;
- (2) if  $B \in \mathcal{B}$ , then
 
$$\{Z \setminus D : D \in \mathcal{D}_B\} \subseteq \mathcal{D}_{X \setminus B};$$
- (3) if  $B_1, B_2, \dots \in \mathcal{B}$ , then

$$\left\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathcal{D}_{B_n} \right\} \subseteq \mathcal{D}_{\bigcup_{n=1}^{\infty} B_n};$$

*and a collection  $\{S_{\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}$  of pairwise disjoint non-empty sets, bijectively indexed by a collection  $\mathbb{U}$  of ultrafilters in  $\mathcal{B}$  with c.i.p., where the sets  $X, S_{\mathcal{U}}$  for any  $\mathcal{U} \in \mathbb{U}$ , and  $Z$  are pairwise disjoint, such that*

$$Y = X \cup \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup Z,$$

$$\mathcal{C} = \left\{ B \cup \bigcup_{B \in \mathcal{B}} S_{\mathcal{U}} \cup D : B \in \mathcal{B} \text{ and } D \in \mathcal{D}_B \right\},$$

and  $\lambda: \mathcal{C} \rightarrow [0, \infty]$  is given by

$$\lambda\left(B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D\right) = \mu(B)$$

for each  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ .

*Proof.* Suppose that  $Y$ ,  $\mathcal{C}$  and  $\lambda$  are defined as in the statement of the theorem. We show that  $(Y, \mathcal{C}, \lambda)$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded. First, we verify that  $\mathcal{C}$  is a  $\sigma$ -algebra on  $Y$ . By (1) we have  $\emptyset \in \mathcal{C}$ . Let  $C \in \mathcal{C}$ . We show that  $Y \setminus C \in \mathcal{C}$ . Let

$$C = B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D$$

for some  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ . Note that for each  $\mathcal{U} \in \mathbb{U}$  we have  $B \notin \mathcal{U}$  if and only if  $X \setminus B \in \mathcal{U}$ ; this is because if  $B \notin \mathcal{U}$  then  $X \setminus B \in \mathcal{U}$  by Lemma 2.1; the converse is trivial. Therefore

$$\begin{aligned} Y \setminus C &= \left(X \cup \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup Z\right) \setminus \left(B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D\right) \\ &= (X \setminus B) \cup \left(\bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \setminus \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}}\right) \cup (Z \setminus D) \\ &= (X \setminus B) \cup \bigcup_{B \notin \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup (Z \setminus D) \\ &= (X \setminus B) \cup \bigcup_{X \setminus B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup (Z \setminus D). \end{aligned}$$

By (2) we have  $Z \setminus D \in \mathcal{D}_{X \setminus B}$ . Thus  $Y \setminus C \in \mathcal{C}$ . Now, to show that  $\mathcal{C}$  is closed under countable unions, let  $C_1, C_2, \dots \in \mathcal{C}$ . Then

$$C_n = B_n \cup \bigcup_{B_n \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D_n$$

where  $B_n \in \mathcal{B}$  and  $D_n \in \mathcal{D}_{B_n}$  for each  $n \in \mathbb{N}$ . Using Lemma 2.1, we have

$$\begin{aligned} \bigcup_{n=1}^{\infty} C_n &= \bigcup_{n=1}^{\infty} \left(B_n \cup \bigcup_{B_n \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D_n\right) \\ &= \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{n=1}^{\infty} \bigcup_{B_n \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup \bigcup_{n=1}^{\infty} D_n \\ &= \bigcup_{n=1}^{\infty} B_n \cup \bigcup_{\bigcup_{n=1}^{\infty} B_n \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup \bigcup_{n=1}^{\infty} D_n. \end{aligned}$$

By (3) we have

$$\bigcup_{n=1}^{\infty} D_n \in \mathcal{D}_{\bigcup_{n=1}^{\infty} B_n}.$$

Thus

$$\bigcup_{n=1}^{\infty} C_n \in \mathcal{C}.$$

This shows that  $\mathcal{C}$  is a  $\sigma$ -algebra on  $Y$ . Next, we show that  $\lambda$  is a measure on  $\mathcal{C}$ . Note that  $\lambda(\emptyset) = 0$ . If  $C_1, C_2, \dots \in \mathcal{C}$  are disjoint, then using the above results and notation we have

$$\lambda\left(\bigcup_{n=1}^{\infty} C_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) = \sum_{n=1}^{\infty} \lambda(C_n).$$

This shows that  $(Y, \mathcal{C}, \lambda)$  is a measure space. Now we show that  $(X, \mathcal{B}, \mu)$  is embedded in  $(Y, \mathcal{C}, \lambda)$ . Obviously, by our definitions we have  $X \subseteq Y$  and  $C \cap X \in \mathcal{B}$  for each  $C \in \mathcal{C}$ . Conversely, for each  $B \in \mathcal{B}$ , since by our assumption  $\mathcal{D}_B$  is non-empty, we have  $B = C \cap X$  for some  $C \in \mathcal{C}$ . Thus

$$\mathcal{B} = \{C \cap X : C \in \mathcal{C}\}.$$

Also, it is obvious that  $\lambda(C) = \mu(C \cap X)$  for each  $C \in \mathcal{C}$ . Therefore  $(X, \mathcal{B}, \mu)$  is embedded in  $(Y, \mathcal{C}, \lambda)$ .

Now, suppose that  $(Y, \mathcal{C}, \lambda)$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded. We show that  $(Y, \mathcal{C}, \lambda)$  can be constructed as in the previous part. Note that  $X \subseteq Y$ . Define

$$Z = \{p \in Y \setminus X : p \in C \subseteq Y \setminus X \text{ for some } C \in \mathcal{C}\}$$

and

$$\mathcal{D} = \{C \cap Z : C \in \mathcal{C}\}.$$

Then obviously  $(Z, \mathcal{D})$  is a measurable space. Define

$$\mathcal{D}_B = \{C \cap Z : C \in \mathcal{C} \text{ and } C \cap X = B\}$$

for each  $B \in \mathcal{B}$ . Obviously  $\mathcal{D}_B \subseteq \mathcal{D}$  and  $\mathcal{D}_B$  is non-empty for each  $B \in \mathcal{B}$ . We verify that conditions (1)–(3) of the theorem hold. Condition (1) holds trivially. To show condition (2), note that if  $D \in \mathcal{D}_B$  for some  $B \in \mathcal{B}$  then  $D = C \cap Z$ , where  $C \in \mathcal{C}$  and  $C \cap X = B$ . Thus

$$Z \setminus D = Z \setminus (C \cap Z) = Z \cap (Y \setminus C).$$

Now, since

$$(Y \setminus C) \cap X = X \setminus (C \cap X) = X \setminus B$$

we have  $Z \setminus D \in \mathcal{D}_{X \setminus B}$ . Therefore

$$\{Z \setminus D : D \in \mathcal{D}_B\} \subseteq \mathcal{D}_{X \setminus B}.$$

To show condition (3), let  $B_n \in \mathcal{B}$  and  $D_n \in \mathcal{D}_{B_n}$  for each  $n \in \mathbb{N}$ . Then  $D_n = C_n \cap Z$  where  $C_n \in \mathcal{C}$  and  $C_n \cap X = B_n$  for each  $n \in \mathbb{N}$ . We have

$$\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} (Z \cap C_n) = Z \cap \bigcup_{n=1}^{\infty} C_n$$

and

$$X \cap \bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (X \cap C_n) = \bigcup_{n=1}^{\infty} B_n$$

where

$$\bigcup_{n=1}^{\infty} C_n \in \mathcal{C}.$$

Thus

$$\bigcup_{n=1}^{\infty} D_n \in \mathcal{D}_{\bigcup_{n=1}^{\infty} B_n},$$

i.e.,

$$\left\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathcal{D}_{B_n} \right\} \subseteq \mathcal{D}_{\bigcup_{n=1}^{\infty} B_n}.$$

This shows conditions (1)–(3). For each  $p \in (Y \setminus X) \setminus Z$  let

$$\mathcal{U}_p = \{C \cap X : p \in C \in \mathcal{C}\}.$$

**Claim 1.** *For each  $p \in (Y \setminus X) \setminus Z$  the set  $\mathcal{U}_p$  is an ultrafilter in  $\mathcal{B}$  which has c.i.p.*

*Proof of Claim 1.* First note that

$$(2.2) \quad (Y \setminus X) \setminus Z = \{y \in Y \setminus X : C \cap X \neq \emptyset \text{ for each } C \in \mathcal{C} \text{ with } y \in C\}.$$

Let  $p \in (Y \setminus X) \setminus Z$ . By (2.2) we have  $\emptyset \notin \mathcal{U}_p$ . It is obvious that  $\emptyset \neq \mathcal{U}_p \subseteq \mathcal{B}$  and that  $\mathcal{U}_p$  is closed under finite intersections. Now, suppose that  $U \subseteq B$  for some  $U \in \mathcal{U}_p$  and  $B \in \mathcal{B}$ . Then  $U = C \cap X$  for some  $C \in \mathcal{C}$  such that  $p \in C$ , and  $B = G \cap X$  for some  $G \in \mathcal{C}$ . If  $p \notin G$  then  $p \in Y \setminus G \in \mathcal{C}$ . Thus  $p \in C \cap (Y \setminus G) \in \mathcal{C}$  and therefore by (2.2) and the choice of  $p$  the set  $C \cap (Y \setminus G) \cap X$  is non-empty. But this is a contradiction, as

$$C \cap X \cap (Y \setminus G) = U \cap (Y \setminus G) \subseteq B \cap (Y \setminus G) = (G \cap X) \cap (Y \setminus G) = \emptyset.$$

Thus  $p \in G$  and therefore  $B = G \cap X \in \mathcal{U}_p$ . This shows that  $\mathcal{U}_p$  is a filter in  $\mathcal{B}$ . To show that  $\mathcal{U}_p$  is an ultrafilter, let  $B \in \mathcal{B}$  be such that  $B \cap U$  is non-empty for each  $U \in \mathcal{U}_p$ . Let  $B = C \cap X$  for some  $C \in \mathcal{C}$ . If  $p \notin C$  then  $p \in Y \setminus C$  and thus  $(Y \setminus C) \cap X \in \mathcal{U}_p$ , which is not possible, as  $(Y \setminus C) \cap X$  misses  $B$ . Therefore  $p \in C$  and thus  $B = C \cap X \in \mathcal{U}_p$ . To show that  $\mathcal{U}_p$  has c.i.p., let  $U_1, U_2, \dots \in \mathcal{U}_p$ . Then  $U_n = C_n \cap X$  where  $p \in C_n \in \mathcal{C}$  for each  $n \in \mathbb{N}$ . Then

$$p \in \bigcap_{n=1}^{\infty} C_n \in \mathcal{C}$$

and thus

$$\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (X \cap C_n) = X \cap \bigcap_{n=1}^{\infty} C_n \in \mathcal{U}_p.$$

Therefore

$$\bigcap_{n=1}^{\infty} U_n \neq \emptyset.$$

This proves the claim.

Let

$$\mathbb{U} = \{\mathcal{U}_p : p \in (Y \setminus X) \setminus Z\}.$$

Then  $\mathbb{U}$  is a collection of ultrafilters in  $\mathcal{B}$  with c.i.p. For each  $\mathcal{U} \in \mathbb{U}$  define

$$S_{\mathcal{U}} = \{p \in (Y \setminus X) \setminus Z : \mathcal{U}_p = \mathcal{U}\}.$$

Note that  $S_{\mathcal{U}}$ , for each  $\mathcal{U} \in \mathbb{U}$ , is non-empty, as  $\mathcal{U} = \mathcal{U}_p$  for some  $p \in (Y \setminus X) \setminus Z$  and thus  $p \in S_{\mathcal{U}}$ . Also, for any distinct  $\mathcal{U}, \mathcal{V} \in \mathbb{U}$  we have  $S_{\mathcal{U}} \cap S_{\mathcal{V}} = \emptyset$ , as  $p \in S_{\mathcal{U}} \cap S_{\mathcal{V}}$  implies that  $\mathcal{U} = \mathcal{U}_p = \mathcal{V}$ . Thus

$$\{S_{\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}$$

is a bijectively indexed collection of pairwise disjoint non-empty sets. Note that by our definitions the sets  $X, S_{\mathcal{U}}$  where  $\mathcal{U} \in \mathbb{U}$  and  $Z$  are pairwise disjoint. Let

$$Y' = X \cup \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup Z$$

$$\mathcal{C}' = \left\{ B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D : B \in \mathcal{B} \text{ and } D \in \mathcal{D}_B \right\}$$

and  $\lambda' : \mathcal{C}' \rightarrow [0, \infty]$  be given by

$$\lambda' \left( B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D \right) = \mu(B)$$

where  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ . By the first part we know that  $(Y', \mathcal{C}', \lambda')$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded. We verify that

$$(Y, \mathcal{C}, \lambda) = (Y', \mathcal{C}', \lambda').$$

By our definition it is obvious that  $Y' \subseteq Y$ . To show the reverse inclusion, let  $p \in Y$ . If either  $p \in X$  or  $p \in Z$  then  $p \in Y'$ . If  $p \in (Y \setminus X) \setminus Z$ , then since  $\mathcal{U}_p \in \mathbb{U}$  and by our definition  $p \in S_{\mathcal{U}_p}$ , we have  $p \in Y'$ . Thus  $Y \subseteq Y'$  and therefore  $Y = Y'$ . Next, we verify that  $\mathcal{C} = \mathcal{C}'$ .

**Claim 2.** *Let  $C \in \mathcal{C}$  and  $B = C \cap X$ . Then*

$$\bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} = C \cap ((Y \setminus X) \setminus Z).$$

*Proof of Claim 2.* Suppose that  $p \in S_{\mathcal{U}}$  for some  $\mathcal{U} \in \mathbb{U}$  such that  $B \in \mathcal{U}$ . If  $p \notin C$  then  $p \in Y \setminus C \in \mathcal{C}$  and thus

$$(Y \setminus C) \cap X \in \mathcal{U}_p = \mathcal{U}.$$

But this is not possible, as  $(Y \setminus C) \cap X$  misses  $C \cap X = B$ . Thus  $p \in C$ . Also, since  $S_{\mathcal{U}} \subseteq (Y \setminus X) \setminus Z$  it is obvious that  $p \in (Y \setminus X) \setminus Z$ . To show the reverse inclusion, note that for each  $p \in C \cap ((Y \setminus X) \setminus Z)$  since  $p \in C$  we have  $B = C \cap X \in \mathcal{U}_p$  and  $p \in S_{\mathcal{U}_p}$ . This proves the claim.

Now, let  $C' \in \mathcal{C}'$ . Then

$$C' = B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D.$$

for some  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ . Thus, by the way we have defined  $\mathcal{D}_B$  we have  $D = C \cap Z$ , for some  $C \in \mathcal{C}$  such that  $C \cap X = B$ . By Claim 2 we have

$$C' = B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D = (C \cap X) \cup (C \cap ((Y \setminus X) \setminus Z)) \cup (C \cap Z) = C \in \mathcal{C}$$

Therefore  $\mathcal{C}' \subseteq \mathcal{C}$ . To show the reverse inclusion let  $C \in \mathcal{C}$ . Let  $B = C \cap X \in \mathcal{B}$  and  $D = C \cap Z$ . Then  $D \in \mathcal{D}_B$ , by the way we have defined  $\mathcal{D}_B$ , and thus by Claim 2 we have

$$C = (C \cap X) \cup (C \cap ((Y \setminus X) \setminus Z)) \cup (C \cap Z) = B \cup \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup D \in \mathcal{C}'.$$

Therefore  $\mathcal{C} \subseteq \mathcal{C}'$ , which together with the above shows that  $\mathcal{C} = \mathcal{C}'$ . The fact that  $\lambda = \lambda'$  is trivial, as by the above for each  $C \in \mathcal{C}$  we have

$$\lambda(C) = \mu(C \cap X) = \lambda'(C).$$

This completes the proof.  $\square$

Let  $(Y, \mathcal{C}, \lambda)$  be a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded. Assume the representation and notation given in Theorem 2.2. Then

$$Y = X \cup \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} \cup Z$$

where (by the proof of Theorem 2.2)

$$Z = \{p \in Y \setminus X : p \in C \subseteq Y \setminus X \text{ for some } C \in \mathcal{C}\}.$$

Thus

$$\begin{aligned} \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} &= (Y \setminus X) \setminus Z \\ &= \{p \in Y \setminus X : C \cap X \neq \emptyset \text{ for each } C \in \mathcal{C} \text{ with } p \in C\}. \end{aligned}$$

We verify that

$$\begin{aligned} \bigcup_{\mathcal{U} \in \mathbb{U} \text{ is free}} S_{\mathcal{U}} &= \\ \{p \in (Y \setminus X) \setminus Z : p \text{ is separated from each } x \in X \text{ by sets in } \mathcal{C}\} \end{aligned}$$

and consequently

$$\begin{aligned} \bigcup_{\mathcal{U} \in \mathbb{U} \text{ is fixed}} S_{\mathcal{U}} &= \\ \{p \in (Y \setminus X) \setminus Z : p \text{ is not separated from some } x \in X \text{ by sets in } \mathcal{C}\}. \end{aligned}$$

To show this, let  $p \in S_{\mathcal{U}}$  for some free  $\mathcal{U} \in \mathbb{U}$ . Let  $x \in X$ . Since  $\mathcal{U}$  is free, we have  $x \notin U$  for some  $U \in \mathcal{U}$ . Let  $D \in \mathcal{D}_U$ . Then

$$C = U \cup \bigcup_{U \in \mathcal{V} \in \mathbb{U}} S_{\mathcal{V}} \cup D \in \mathcal{C}$$

is such that  $p \in C$  and  $x \notin C$ . Conversely, let  $p \in (Y \setminus X) \setminus Z$  be such that it can be separated from each  $x \in X$  by a measurable set in  $\mathcal{C}$ . Let  $\mathcal{U} \in \mathbb{U}$  be such that  $p \in S_{\mathcal{U}}$ . Suppose that  $\mathcal{U}$  is not free. Let  $x \in \bigcap \mathcal{U}$ . Let

$$C = B \cup \bigcup_{B \in \mathcal{V} \in \mathbb{U}} S_{\mathcal{V}} \cup D \in \mathcal{C},$$

where  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ , be such that  $p \in C$  and  $x \notin C$ . Then, since  $p \in S_{\mathcal{U}}$ , we have  $B \in \mathcal{U}$ , and thus  $x \in B$ , which is not possible, as  $B \subseteq C$ . Therefore  $\mathcal{U}$  is free.

Thus, in the absence of free ultrafilters in  $\mathcal{B}$ , each  $p \in Y \setminus X$  (depending on whether  $p \in Z$  or  $p \notin Z$ ) either “separates” from the whole  $X$  by a (null) set in  $\mathcal{C}$ , or tightly “sticks” to some point  $x$  of  $X$  so that it cannot be separated from  $x$  by any measurable set in  $\mathcal{C}$ . In the next section we restrict our attention to measure spaces  $(X, \mathcal{B}, \mu)$  having no free ultrafilter in  $\mathcal{B}$  with c.i.p. As we will see, this assumption considerably simplifies our construction.



### 3. THE CASE OF MEASURE SPACES $(X, \mathcal{B}, \mu)$ WITH NO FREE ULTRAFILTER IN $\mathcal{B}$ HAVING C.I.P.

In this section we show that for certain classes of measure spaces  $(X, \mathcal{B}, \mu)$ , the structure of measure spaces  $(Y, \mathcal{C}, \lambda)$  in which  $(X, \mathcal{B}, \mu)$  is embedded is expressible in a simpler way: They are simply obtained by “blowing” certain points of  $X$  up and “pasting” a certain measurable space to  $X$  in a certain way. This we prove in the next theorem. Examples of measure spaces satisfying this assumption are given in the next section.

**Theorem 3.1.** *Let  $(X, \mathcal{B}, \mu)$  be a measure space. Suppose that the points of  $X$  are separated by measurable sets in  $\mathcal{B}$  and that there is no free ultrafilter in  $\mathcal{B}$  with c.i.p. Then  $(Y, \mathcal{C}, \lambda)$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded if and only if there exists a measurable space  $(Z, \mathcal{D})$ , a collection  $\{\mathcal{D}_B : B \in \mathcal{B}\}$  of non-empty subsets of  $\mathcal{D}$  such that*

- (1)  $\emptyset \in \mathcal{D}_\emptyset$ ;
- (2) if  $B \in \mathcal{B}$ , then

$$\{Z \setminus D : D \in \mathcal{D}_B\} \subseteq \mathcal{D}_{X \setminus B};$$

- (3) if  $B_1, B_2, \dots \in \mathcal{B}$ , then

$$\left\{ \bigcup_{n=1}^{\infty} D_n : D_n \in \mathcal{D}_{B_n} \right\} \subseteq \mathcal{D}_{\bigcup_{n=1}^{\infty} B_n};$$

and a collection  $\{T_u : u \in U\}$  of pairwise disjoint non-empty sets, bijectively indexed by a subset  $U$  of  $X$ , where the sets  $X, T_u$  for any  $u \in U$ , and  $Z$  are pairwise disjoint, such that

$$Y = X \cup \bigcup_{u \in U} T_u \cup Z,$$

$$\mathcal{C} = \left\{ B \cup \bigcup_{u \in B \cap U} T_u \cup D : B \in \mathcal{B} \text{ and } D \in \mathcal{D}_B \right\}$$

and  $\lambda : \mathcal{C} \rightarrow [0, \infty]$  is given by

$$\lambda\left(B \cup \bigcup_{u \in B \cap U} T_u \cup D\right) = \mu(B)$$

for each  $B \in \mathcal{B}$  and  $D \in \mathcal{D}_B$ .

*Proof.* Suppose that  $(Y, \mathcal{C}, \lambda)$  is a measure space in which  $(X, \mathcal{B}, \mu)$  is embedded. Assume the representation given for  $(Y, \mathcal{C}, \lambda)$  in Theorem 2.2. Assume the notation of Theorem 2.2. By our assumption for each  $\mathcal{U} \in \mathbb{U}$  the set  $\bigcap \mathcal{U}$  is non-empty. Note that  $\bigcap \mathcal{U}$  is a singleton, as if  $x, z \in \bigcap \mathcal{U}$  and  $x \neq z$ , then by our assumption  $x \in B$  and  $z \notin B$  for some  $B \in \mathcal{B}$ . Now  $X \setminus B \in \mathcal{B}$  intersects each element of  $\mathcal{U}$ , thus  $X \setminus B \in \mathcal{U}$ . This contradicts the fact that  $x \notin X \setminus B$ . Let

$$\bigcap \mathcal{U} = \{u_{\mathcal{U}}\}.$$

Define

$$U = \{u_{\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}.$$

**Claim 1.** *For each  $\mathcal{U} \in \mathbb{U}$  we have*

$$\mathcal{U} = \{B \in \mathcal{B} : u_{\mathcal{U}} \in B\}.$$

*Proof of Claim 1.* Let  $\mathcal{U} \in \mathbb{U}$ . Obviously,  $u_{\mathcal{U}} \in B$  for each  $B \in \mathcal{U}$ . Conversely, if  $B \in \mathcal{B}$  is such that  $u_{\mathcal{U}} \in B$  then  $B \in \mathcal{U}$ . As otherwise,  $B \cap G = \emptyset$  for some  $G \in \mathcal{U}$ . Thus  $G \subseteq X \setminus B \in \mathcal{B}$  which implies that  $X \setminus B \in \mathcal{U}$ . This contradicts the fact that  $u_{\mathcal{U}} \notin X \setminus B$  and proves the claim.

For each  $u \in U$  there exists some  $\mathcal{U} \in \mathbb{U}$  such that  $u = u_{\mathcal{U}}$ . Note that by Claim 1 such a  $\mathcal{U}$  is unique; let  $T_u = S_{\mathcal{U}}$ . The collection

$$\{T_u : u \in U\}$$

consists of non-empty sets which are pairwise disjoint and bijectively indexed (as any distinct  $u, v \in U$  are of the form  $u = u_{\mathcal{U}}$  and  $v = u_{\mathcal{V}}$  for some distinct  $\mathcal{U}, \mathcal{V} \in \mathbb{U}$ ).

**Claim 2.** For each  $B \in \mathcal{B}$  we have

$$\bigcup_{u \in B \cap U} T_u = \bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}}.$$

*Proof of Claim 2.* Suppose that  $B \in \mathcal{U} \in \mathbb{U}$ . By Claim 1 we have  $u_{\mathcal{U}} \in B \cap U$ . Note that  $T_{u_{\mathcal{U}}} = S_{\mathcal{U}}$ . To show the reverse inclusion, let  $u \in B \cap U$ . Let  $\mathcal{U} \in \mathbb{U}$  be such that  $u = u_{\mathcal{U}}$ . Then, by our definition  $T_u = S_{\mathcal{U}}$ . Note that by Claim 1 we have  $B \in \mathcal{U}$ . This proves the claim.

Note that if  $B = X$  then Claim 2 implies that

$$\bigcup_{u \in U} T_u = \bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}}.$$

From the above the desired representation of  $(Y, \mathcal{C}, \lambda)$  follows.

Conversely, suppose that  $Y, \mathcal{C}$  and  $\lambda$  are given as in the statement of the theorem. Assume the notation of the theorem. For each  $u \in U$  define

$$\mathcal{U}_u = \{B \in \mathcal{B} : u \in B\}.$$

**Claim 3.** For each  $u \in U$  the set  $\mathcal{U}_u$  is an ultrafilter in  $\mathcal{B}$  with c.i.p.

*Proof of Claim 3.* Let  $u \in U$ . Obviously,  $\emptyset \neq \mathcal{U}_u \subseteq \mathcal{B}$ ,  $\emptyset \notin \mathcal{U}_u$  and  $\mathcal{U}_u$  is closed under finite intersections. Also, note that if  $G \subseteq B$  for some  $G \in \mathcal{U}_u$  and  $B \in \mathcal{B}$ , then  $B \in \mathcal{U}_u$ . Thus  $\mathcal{U}_u$  is a filter in  $\mathcal{B}$ . To show that  $\mathcal{U}_u$  is an ultrafilter, suppose that  $B \in \mathcal{B}$  is such that  $B \cap G$  is non-empty for each  $G \in \mathcal{U}_u$ . If  $B \notin \mathcal{U}_u$  then  $u \notin B$ . Thus  $u \in X \setminus B \in \mathcal{B}$  and  $X \setminus B \in \mathcal{U}_u$ . But  $X \setminus B$  misses  $B$ , which is a contradiction. Therefore  $B \in \mathcal{U}_u$ . The fact that  $\mathcal{U}_u$  has c.i.p. is obvious. This proves the claim.

Let

$$\mathbb{U} = \{\mathcal{U}_u : u \in U\}.$$

Note that each  $\mathcal{U} \in \mathbb{U}$  is of the form  $\mathcal{U}_u$  for some unique  $u \in U$ . This is because if  $\mathcal{U}_u = \mathcal{U}_v$  for some distinct  $u, v \in U$ , then by our assumption  $u \in B$  and  $v \notin B$  for some  $B \in \mathcal{B}$ . Thus  $B \in \mathcal{U}_u \setminus \mathcal{U}_v$ . Therefore  $\mathcal{U}_u \neq \mathcal{U}_v$ , which is a contradiction. For each  $\mathcal{U} \in \mathbb{U}$  define  $S_{\mathcal{U}} = T_u$ , where  $u \in U$  is such that  $\mathcal{U} = \mathcal{U}_u$ . The collection

$$\{S_{\mathcal{U}} : \mathcal{U} \in \mathbb{U}\}$$

consists of non-empty sets which are pairwise disjoint and bijectively indexed (as distinct elements of  $\mathbb{U}$  are assigned to distinct elements of  $U$ ).

**Claim 4.** *For each  $B \in \mathcal{B}$  we have*

$$\bigcup_{B \in \mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} = \bigcup_{u \in B \cap U} T_u.$$

*Proof of Claim 4.* Let  $u \in B \cap U$ . Then, by our definition of  $\mathcal{U}_u$  we have  $B \in \mathcal{U}_u \in \mathbb{U}$ . Also, by our definition  $T_u = S_{\mathcal{U}_u}$ . To show the reverse inclusion, let  $B \in \mathcal{U} \in \mathbb{U}$ . Let  $u \in U$  be such that  $\mathcal{U} = \mathcal{U}_u$ . Then, by our definition  $S_{\mathcal{U}} = T_u$ . But since  $B \in \mathcal{U}_u$ , by our definition we have  $u \in B$ , and thus  $u \in B \cap U$ . This proves the claim.

Note that if  $B = X$  then Claim 4 implies that

$$\bigcup_{\mathcal{U} \in \mathbb{U}} S_{\mathcal{U}} = \bigcup_{u \in U} T_u.$$

From the above and Theorem 2.2 the result follows.  $\square$

#### 4. EXAMPLES OF MEASURE SPACES $(X, \mathcal{B}, \mu)$ WITH NO FREE ULTRAFILTER IN $\mathcal{B}$ HAVING C.I.P.

In this section we give examples of measure spaces  $(X, \mathcal{B}, \mu)$  for which the assumption of Theorem 3.1 holds, i.e., measure spaces  $(X, \mathcal{B}, \mu)$  for which there is no free ultrafilter in  $\mathcal{B}$  with c.i.p. The following gives some equivalent ways to express this condition. The equivalence of conditions (1) and (2) in the following proposition is well known; we include the proof in here for the sake of completeness.

**Proposition 4.1.** *Let  $(X, \mathcal{B})$  be a measurable space. Then the following are equivalent:*

- (1) *There is no free ultrafilter in  $\mathcal{B}$  with c.i.p.*
- (2) *For every  $\{0, 1\}$ -valued measure  $\mu$  on  $\mathcal{B}$  whose null sets cover  $X$  we have  $\mu \equiv 0$ .*
- (3) *For every  $\{0, 1\}$ -valued measure  $\mu$  on  $\mathcal{B}$  whose null sets cover  $X$ , if  $\mathcal{C} \subseteq \mathcal{B}$  is non-empty and such that  $\bigcup \mathcal{C} \in \mathcal{B}$ , then*

$$\mu\left(\bigcup \mathcal{C}\right) = \sup_{C \in \mathcal{C}} \mu(C).$$

*Proof.* That (2) implies (3) is trivial. (1) *implies* (2). Let  $\mu$  be a non-trivial  $\{0, 1\}$ -valued measure on  $\mathcal{B}$  whose null sets cover  $X$ . Define

$$\mathcal{F} = \{B \in \mathcal{B} : \mu(B) = 1\}.$$

We show that  $\mathcal{F}$  is a free ultrafilter in  $\mathcal{B}$  with c.i.p. If  $F, G \in \mathcal{F}$ , then since

$$\mu(F \cup G) = \mu(F \setminus G) + \mu(G)$$

and  $\mu(G) = 1$ , we have  $\mu(F \setminus G) = 0$ . Therefore

$$\mu(F \cap G) = \mu(F \setminus G) + \mu(F \cap G) = \mu(F) = 1$$

and thus  $F \cap G \in \mathcal{F}$ . Obviously, if  $F \subseteq B$  for some  $F \in \mathcal{F}$  and  $B \in \mathcal{B}$ , then  $B \in \mathcal{F}$ . Therefore,  $\mathcal{F}$  is a filter in  $\mathcal{B}$ . To show that  $\mathcal{F}$  is an ultrafilter, let  $B \in \mathcal{B}$  be such that  $B \cap F$  is non-empty for each  $F \in \mathcal{F}$ . If  $B \notin \mathcal{F}$ , then  $\mu(B) = 0$ , and thus, since  $\mu(X \setminus B) = 1$ , we have  $X \setminus B \in \mathcal{F}$ . But this is not possible, as  $B$  misses

$X \setminus B$ . Therefore  $B \in \mathcal{F}$ . To show that  $\mathcal{F}$  has c.i.p., let  $F_1, F_2, \dots \in \mathcal{F}$ . Without any loss of generality we may assume that  $F_1 \supseteq F_2 \supseteq \dots$ . If

$$\bigcap_{n=1}^{\infty} F_n \notin \mathcal{F}$$

then

$$\mu\left(\bigcap_{n=1}^{\infty} F_n\right) = 0.$$

Thus (with the empty intersection interpreted as  $X$ ) we have

$$(4.1) \quad 1 = \mu\left(X \setminus \bigcap_{n=1}^{\infty} F_n\right) = \mu\left(\bigcup_{n=1}^{\infty} (F_{n-1} \setminus F_n)\right) = \sum_{n=1}^{\infty} \mu(F_{n-1} \setminus F_n).$$

Now, each  $G_n = F_{n-1} \setminus F_n$ , where  $n \in \mathbb{N}$ , misses  $F_n \in \mathcal{F}$  and thus  $G_n \notin \mathcal{F}$ ; therefore  $\mu(G_n) = 0$ . This contradicts (4.1) and shows that

$$\bigcap_{n=1}^{\infty} F_n \in \mathcal{F}.$$

To show that  $\mathcal{F}$  is free, let  $x \in X$ . By our assumption  $x \in B$  for some  $B \in \mathcal{B}$  such that  $\mu(B) = 0$ . Thus  $\mu(X \setminus B) = 1$ . But  $X \setminus B \in \mathcal{F}$  and  $x \notin X \setminus B$ . Therefore  $\bigcap \mathcal{F} = \emptyset$ .

(3) *implies* (1). Suppose that there exists a free ultrafilter  $\mathcal{F}$  in  $\mathcal{B}$  with c.i.p. Define  $\mu : \mathcal{B} \rightarrow \{0, 1\}$  such that  $\mu(B) = 1$  if  $B \in \mathcal{F}$  and  $\mu(B) = 0$  if  $B \notin \mathcal{F}$ . To show that  $\mu$  is a measure, first note that  $\mu(\emptyset) = 0$ . Let  $B_1, B_2, \dots \in \mathcal{B}$  be pairwise disjoint. Suppose that

$$\bigcup_{n=1}^{\infty} B_n \notin \mathcal{F}.$$

Then  $B_n \notin \mathcal{F}$  for some  $n \in \mathbb{N}$ . Therefore

$$(4.2) \quad \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

as each side is identical to 0. Suppose that

$$\bigcup_{n=1}^{\infty} B_n \in \mathcal{F}.$$

By Lemma 2.1 this implies that  $B_n \in \mathcal{F}$  for some  $n \in \mathbb{N}$ . Note that for each  $n \neq i \in \mathbb{N}$ , since  $B_i \cap B_n = \emptyset$  we have  $B_i \notin \mathcal{F}$ . Thus (4.2) holds, as in this case each side is identical to 1. This shows that  $\mu$  is a measure. Since  $\mathcal{F}$  is free, for each  $x \in X$  there exists some  $F \in \mathcal{F}$  such that  $x \notin F$ . Thus  $x \in X \setminus F$ , and since  $X \setminus F \notin \mathcal{F}$  we have  $\mu(X \setminus F) = 0$ . Therefore the null sets of  $\mathcal{B}$  cover  $X$ . Now, let

$$\mathcal{C} = \{X \setminus F : F \in \mathcal{F}\}.$$

Then since  $\mathcal{F}$  is free, we have  $\bigcup \mathcal{C} = X$ , and therefore by our assumption

$$\sup_{F \in \mathcal{F}} \mu(X \setminus F) = \sup_{C \in \mathcal{C}} \mu(C) = \mu\left(\bigcup \mathcal{C}\right) = \mu(X) = 1.$$

But this is not possible, as if  $F \in \mathcal{F}$  then  $X \setminus F \notin \mathcal{F}$ , as it misses  $F$ , and thus  $\mu(X \setminus F) = 0$ .  $\square$

**Theorem 4.2.** *Let  $(X, \mathcal{B})$  be a measurable space. If  $X$  is countable then there is no free ultrafilter in  $\mathcal{B}$  with c.i.p.*

*Proof.* Let  $X = \{x_1, x_2, \dots\}$  and let  $\mathcal{F}$  be a free ultrafilter in  $\mathcal{B}$ . Since  $\mathcal{F}$  is free for each  $n \in \mathbb{N}$  there is some  $F_n \in \mathcal{F}$  such that  $x_n \notin F_n$ . Then

$$\bigcap_{n=1}^{\infty} F_n = \emptyset$$

and thus  $\mathcal{F}$  does not have c.i.p. □

**Theorem 4.3.** *Let  $(Y, \mathcal{C})$  be a measurable space. Let  $X \in \mathcal{C}$  and*

$$\mathcal{B} = \{C \in \mathcal{C} : C \subseteq X\},$$

*i.e.,  $X \in \mathcal{C}$  and  $(X, \mathcal{B})$  is embedded in  $(Y, \mathcal{C})$ . If there is no free ultrafilter in  $\mathcal{C}$  with c.i.p. then there is no free ultrafilter in  $\mathcal{B}$  with c.i.p.*

*Proof.* Simply note that if  $\mathcal{F}$  is a free ultrafilter in  $\mathcal{B}$  with c.i.p., then

$$\mathcal{G} = \{G \in \mathcal{C} : G \supseteq F \text{ for some } F \in \mathcal{F}\}$$

is a filter in  $\mathcal{C}$  which is an ultrafilter (as if  $C \in \mathcal{C}$  meets each  $G \in \mathcal{G}$  then, since  $\mathcal{F} \subseteq \mathcal{G}$ , the set  $(X \cap C) \cap F = C \cap F$  is non-empty for each  $F \in \mathcal{F}$  and thus  $X \cap C \in \mathcal{F}$  which implies that  $C \in \mathcal{G}$ , as  $X \cap C \subseteq C$ ) and it is free (as  $\mathcal{F} \subseteq \mathcal{G}$  and thus  $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$  and  $\mathcal{F}$  is free) and has c.i.p. (as  $\mathcal{F}$  has, and if  $G_1, G_2, \dots \in \mathcal{G}$  then

$$\bigcap_{n=1}^{\infty} G_n \supseteq \bigcap_{n=1}^{\infty} F_n,$$

where for each  $n \in \mathbb{N}$ , the element  $F_n \in \mathcal{F}$  is such that  $F_n \subseteq G_n$ ). □

**Theorem 4.4.** *Let  $(X, \mathcal{B}) \subseteq (Y, \mathcal{C})$  and let  $Y \setminus X$  be countable. If there is no free ultrafilter in  $\mathcal{B}$  with c.i.p. then there is no free ultrafilter in  $\mathcal{C}$  with c.i.p.*

*Proof.* Let

$$Y \setminus X = \{y_1, y_2, \dots\}.$$

Let  $\mathcal{H}$  be a free ultrafilter in  $\mathcal{C}$  with c.i.p. Since  $\mathcal{H}$  is free, for any  $n \in \mathbb{N}$  there exists some  $H_n \in \mathcal{H}$  such that  $y_n \notin H_n$ . Let

$$\mathcal{A} = \{H \cap X : H \in \mathcal{H}\} \subseteq \mathcal{B}.$$

Note that  $\emptyset \notin \mathcal{A}$ , as otherwise  $H \cap X = \emptyset$  for some  $H \in \mathcal{H}$ . Now

$$H \cap \bigcap_{n=1}^{\infty} H_n = \emptyset,$$

as

$$H \cap \bigcap_{n=1}^{\infty} H_n \subseteq (X \cap H) \cup \left( (Y \setminus X) \cap \bigcap_{n=1}^{\infty} H_n \right) = \emptyset$$

contradicting the fact that  $\mathcal{H}$  has c.i.p. Thus  $\mathcal{A}$  is a filter-base in  $\mathcal{B}$ , as  $\mathcal{A}$  is closed under finite intersections. Let  $\mathcal{F}$  be an ultrafilter in  $\mathcal{B}$  such that  $\mathcal{A} \subseteq \mathcal{F}$ . Then since

$$\bigcap \mathcal{F} \subseteq \bigcap \mathcal{A} = \bigcap \mathcal{H} \cap X$$

(and  $\mathcal{H}$  is free),  $\mathcal{F}$  is free. To show that  $\mathcal{F}$  has c.i.p., let  $F_1, F_2, \dots \in \mathcal{F}$ . Let  $F_n = C_n \cap X$  where  $C_n \in \mathcal{C}$  for each  $n \in \mathbb{N}$ . Let  $n \in \mathbb{N}$ . For each  $H \in \mathcal{H}$ , since  $H \cap X \in \mathcal{A} \subseteq \mathcal{F}$  we have

$$C_n \cap H \cap X = F_n \cap H \cap X \in \mathcal{F}.$$

Therefore  $H \cap C_n$  is non-empty and thus (since  $\mathcal{H}$  is an ultrafilter)  $C_n \in \mathcal{H}$ . Now, since

$$\begin{aligned} \bigcap_{n=1}^{\infty} C_n \cap \bigcap_{n=1}^{\infty} H_n &\subseteq \left( X \cap \bigcap_{n=1}^{\infty} C_n \right) \cup \left( (Y \setminus X) \cap \bigcap_{n=1}^{\infty} H_n \right) \\ &= \bigcap_{n=1}^{\infty} (X \cap C_n) = \bigcap_{n=1}^{\infty} F_n \end{aligned}$$

and  $\mathcal{H}$  has c.i.p., we have

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Therefore  $\mathcal{F}$  has c.i.p. □

If  $(X, \mathcal{B})$  and  $(Y, \mathcal{C})$  are measurable spaces, we denote by  $\mathcal{B} \times \mathcal{C}$  the smallest  $\sigma$ -algebra on  $X \times Y$  containing the set

$$\{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

of all measurable rectangles in  $X \times Y$ .

**Theorem 4.5.** *Let  $(X, \mathcal{B})$  and  $(Y, \mathcal{C})$  be measurable spaces such that in each of them singletons are measurable. Then the following are equivalent:*

- (1) *There is no free ultrafilter in  $\mathcal{B} \times \mathcal{C}$  with c.i.p.*
- (2) *Neither there is a free ultrafilter in  $\mathcal{B}$  with c.i.p. nor there is a free ultrafilter in  $\mathcal{C}$  with c.i.p.*

*Proof.* (1) *implies* (2). Let  $\mathcal{F}$  be a free ultrafilter in  $\mathcal{B}$  with c.i.p. Fix some  $y \in Y$  and let

$$\mathcal{A} = \{F \times \{y\} : F \in \mathcal{F}\}.$$

Then  $\mathcal{A}$  is a filter-base in  $\mathcal{B} \times \mathcal{C}$ , as  $\emptyset \notin \mathcal{A}$  and  $\mathcal{A}$  is closed under finite intersections. Let  $\mathcal{H}$  be an ultrafilter in  $\mathcal{B} \times \mathcal{C}$  such that  $\mathcal{A} \subseteq \mathcal{H}$ . Then

$$\bigcap \mathcal{H} \subseteq \bigcap \mathcal{A} = \left( \bigcap \mathcal{F} \right) \times \{y\} = \emptyset$$

and  $\mathcal{H}$  is also free. We verify that  $\mathcal{H}$  has c.i.p. Let  $H_1, H_2, \dots \in \mathcal{H}$ . Let

$$H_n^y = \{x \in X : (x, y) \in H_n\}$$

for each  $n \in \mathbb{N}$ . Then  $H_n^y$ , for each  $n \in \mathbb{N}$ , being the  $y$ -section of the measurable set  $H_n$  of  $\mathcal{B} \times \mathcal{C}$  is measurable in  $X$ . Note that for each  $n \in \mathbb{N}$  and  $F \in \mathcal{F}$ , since  $F \times \{y\} \in \mathcal{H}$ , we have

$$H_n \cap (F \times \{y\}) \neq \emptyset,$$

and thus  $H_n^y \cap F$  is non-empty. Therefore, since  $\mathcal{F}$  is an ultrafilter, we have  $H_n^y \in \mathcal{F}$  for each  $n \in \mathbb{N}$ . Since  $\mathcal{F}$  has c.i.p., we have

$$\bigcap_{n=1}^{\infty} H_n^y \neq \emptyset.$$

Now since

$$\left(\bigcap_{n=1}^{\infty} H_n^y\right) \times \{y\} \subseteq \bigcap_{n=1}^{\infty} H_n$$

it follows that

$$\bigcap_{n=1}^{\infty} H_n \neq \emptyset.$$

A similar argument can be used in the case when there is a free ultrafilter in  $\mathcal{C}$  with c.i.p.

(2) *implies* (1). Let  $\mathcal{H}$  be a free ultrafilter in  $\mathcal{B} \times \mathcal{C}$  with c.i.p. Let

$$\mathcal{A} = \{B \in \mathcal{B} : B \times C \in \mathcal{H} \text{ for some } C \in \mathcal{C}\}.$$

Then  $\mathcal{A}$  is a filter-base in  $\mathcal{B}$ , as  $\emptyset \notin \mathcal{A}$  and it is closed under finite intersections. Let  $\mathcal{F}$  be an ultrafilter in  $\mathcal{B}$  such that  $\mathcal{A} \subseteq \mathcal{F}$ .

**Claim.** For each  $F \in \mathcal{F}$  we have  $F \times Y \in \mathcal{H}$ .

*Proof of the claim.* Otherwise, if  $F \times Y \notin \mathcal{H}$  for some  $F \in \mathcal{F}$ , then since  $\mathcal{H}$  is an ultrafilter, we have  $(F \times Y) \cap H = \emptyset$  for some  $H \in \mathcal{H}$ . Thus  $H \subseteq (X \setminus F) \times Y$  and therefore  $(X \setminus F) \times Y \in \mathcal{H}$ . But this implies that  $X \setminus F \in \mathcal{A}$ , and thus  $X \setminus F \in \mathcal{F}$ , which is not possible, as it misses  $F \in \mathcal{F}$ .

Now, we show that  $\mathcal{F}$  has c.i.p. Let  $F_1, F_2, \dots \in \mathcal{F}$ . By the above  $F_n \times Y \in \mathcal{H}$  for each  $n \in \mathbb{N}$  and thus, since  $\mathcal{H}$  has c.i.p.,

$$\bigcap_{n=1}^{\infty} (F_n \times Y) \neq \emptyset.$$

But

$$\bigcap_{n=1}^{\infty} (F_n \times Y) = \left(\bigcap_{n=1}^{\infty} F_n\right) \times Y.$$

Therefore

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

By our assumption  $\mathcal{F}$  is not free, i.e.,  $\bigcap \mathcal{F}$  is non-empty. Let  $p \in \bigcap \mathcal{F}$ . Then  $\{p\} \in \mathcal{B}$  meets each  $F \in \mathcal{F}$ , and thus, since  $\mathcal{F}$  is an ultrafilter we have  $\{p\} \in \mathcal{F}$ . By the claim  $\{p\} \times Y \in \mathcal{H}$ . Similarly, there exists some  $q \in Y$  such that  $X \times \{q\} \in \mathcal{H}$ . Now

$$\{(p, q)\} = (\{p\} \times Y) \cap (X \times \{q\}) \in \mathcal{H}$$

and thus, since  $\{(p, q)\} \cap H$  is non-empty for each  $H \in \mathcal{H}$ , we have  $(p, q) \in \bigcap \mathcal{H}$ , which is a contradiction.  $\square$

Let  $X$  be a completely regular topological space. Recall that the  $\sigma$ -algebra of Baire subsets of  $X$  (denoted by  $\mathcal{B}^*(X)$ ) is the smallest  $\sigma$ -algebra in  $X$  containing  $\mathcal{Z}(X)$ . By a Baire measure on  $X$  we mean a finite measure on  $\mathcal{B}^*(X)$ . The support of a Baire measure  $\mu$  on  $X$  is defined to be the set

$$\{x \in X : \mu(U) > 0 \text{ for every } U \in \text{Coz}(X) \text{ such that } x \in U\}$$

and is denoted by  $\text{supp}(\mu)$ .

The following Lemma is well known. (See [2].)

**Lemma 4.6.** *Let  $X$  be a completely regular topological space. Then the following are equivalent:*

- (1)  *$X$  is realcompact.*
- (2) *Each  $\{0, 1\}$ -valued non-trivial Baire measure on  $X$  has a non-empty support.*
- (3) *Each ultrafilter in  $\mathcal{L}(X)$  with c.i.p. is fixed.*

**Theorem 4.7.** *Let  $(X, \mathcal{O}, \mathcal{B})$  be a first-countable realcompact topological measurable space. Then there is no free ultrafilter in  $\mathcal{B}$  with c.i.p.*

*Proof.* We prove the theorem in two different ways. Our first approach, which is rather direct, is more topological; our second approach makes use of the characterization given in Lemma 4.6.

*First approach.* Suppose to the contrary that there exists a free ultrafilter  $\mathcal{F}$  in  $\mathcal{B}$  with c.i.p. Note that the collection  $\mathcal{B}$  of all measurable sets can be considered as a base for a topology on  $X$ , as it is closed under finite intersections and covers  $X$ . Denote by  $\mathcal{O}_{\mathcal{B}}$  the topology generated by  $\mathcal{B}$  on  $X$ . Since  $(X, \mathcal{O})$  is Hausdorff and  $\mathcal{O} \subseteq \mathcal{B}$ , the topological space  $(X, \mathcal{O}_{\mathcal{B}})$  is Hausdorff and therefore completely regular, as the elements of  $\mathcal{B}$  are closed-open in  $(X, \mathcal{O}_{\mathcal{B}})$ . Let

$$\phi : \beta(X, \mathcal{O}_{\mathcal{B}}) \rightarrow \beta(X, \mathcal{O})$$

continuously extend

$$\text{id}_X : (X, \mathcal{O}_{\mathcal{B}}) \rightarrow (X, \mathcal{O}).$$

Since

$$\{\text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})} F : F \in \mathcal{F}\}$$

has the finite intersection property, as  $\mathcal{F}$  has, by compactness we have

$$G = \bigcap \{\text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})} F : F \in \mathcal{F}\} \neq \emptyset.$$

Let  $p \in G$ . Note that since  $\mathcal{F}$  is free we have  $p \notin X$ , as otherwise

$$p \in X \cap \text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})} F = F$$

for each  $F \in \mathcal{F}$ .

**Claim.** *If  $V$  is an open neighborhood of  $\phi(p)$  in  $\beta(X, \mathcal{O})$  then  $V \cap X \in \mathcal{F}$ .*

*Proof of the claim.* Suppose the contrary, i.e., suppose that  $V \cap X \notin \mathcal{F}$ . Note that  $V \cap X \in \mathcal{O} \subseteq \mathcal{B}$ . Since  $\mathcal{F}$  is an ultrafilter,  $V \cap X \cap F = \emptyset$  for some  $F \in \mathcal{F}$ . Since  $V \cap X \in \mathcal{B}$  and  $F \in \mathcal{B}$ , the sets  $V \cap X$  and  $F$  are closed-open in  $(X, \mathcal{O}_{\mathcal{B}})$ , and therefore

$$\text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})}(V \cap X) \cap \text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})} F = \text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})}(V \cap X \cap F) = \emptyset.$$

By the choice of  $p$  we have

$$p \notin \text{cl}_{\beta(X, \mathcal{O}_{\mathcal{B}})}(V \cap X).$$

Let  $W$  be an open neighborhood of  $p$  in  $\beta(X, \mathcal{O}_{\mathcal{B}})$  such that  $W \cap V \cap X = \emptyset$ . By continuity of  $\phi$  there exists an open neighborhood  $U$  of  $p$  in  $\beta(X, \mathcal{O}_{\mathcal{B}})$  such that  $\phi(U) \subseteq V$ . Then (since  $\phi|_X = \text{id}_X$ ) we have

$$U \cap W \cap X = \phi(U \cap W \cap X) \subseteq \phi(U) \cap W \cap X \subseteq V \cap W \cap X = \emptyset.$$

But this is a contradiction, as  $U \cap W$ , being a non-empty open subset of  $\beta(X, \mathcal{O}_{\mathcal{B}})$ , meets  $X$ . Thus  $V \cap X \in \mathcal{F}$ .



Next, we show that  $\phi(p) \notin X$ . Suppose the contrary. By our assumption, there exists a countable base

$$\{V_n : n \in \mathbb{N}\}$$

at  $\phi(p)$  in  $(X, \mathcal{O})$ . By the claim,  $V_n \in \mathcal{F}$  for each  $n \in \mathbb{N}$ . Now, since  $\mathcal{F}$  has c.i.p., for each  $F \in \mathcal{F}$  we have

$$F \cap \{\phi(p)\} = F \cap \bigcap_{n=1}^{\infty} V_n \neq \emptyset.$$

Thus  $\phi(p) \in \bigcap \mathcal{F}$ , which is a contradiction, as  $\mathcal{F}$  is free. Therefore  $\phi(p) \in \beta(X, \mathcal{O}) \setminus X$ . Since, by our assumption  $(X, \mathcal{O})$  is realcompact, there exists a zero-set  $Z$  in  $\beta(X, \mathcal{O})$  such that  $\phi(p) \in Z$  and  $Z \cap X = \emptyset$ . Let  $Z = f^{-1}(0)$  for some continuous  $f : \beta(X, \mathcal{O}) \rightarrow [0, 1]$ . Now, for each  $n \in \mathbb{N}$  the set  $f^{-1}([0, 1/n))$  is an open neighborhood of  $\phi(p)$  in  $\beta(X, \mathcal{O})$ , thus by the claim

$$f^{-1}([0, 1/n)) \cap X \in \mathcal{F}.$$

Since  $\mathcal{F}$  has c.i.p. we have

$$Z \cap X = f^{-1}(0) \cap X = \bigcap_{n=1}^{\infty} f^{-1}([0, 1/n)) \cap X \neq \emptyset$$

which contradicts the choice of  $Z$ .

*Second approach.* Suppose to the contrary that there exists a free ultrafilter  $\mathcal{F}$  in  $\mathcal{B}$  with c.i.p. Define  $\nu : \mathcal{B} \rightarrow \{0, 1\}$  such that  $\nu(B) = 0$  if  $B \notin \mathcal{F}$  and  $\nu(B) = 1$  if  $B \in \mathcal{F}$ . Then (as in the proof of Proposition 4.1 (3) $\Rightarrow$ (1))  $\nu$  is a measure. Note that  $\mathcal{B}$  contains the set  $\mathcal{B}^*(X)$  of Baire subsets of  $X$  (as each zero-set in  $X$ , being a  $G_\delta$ , is contained in  $\mathcal{B}$ ). Denote

$$\mu = \nu|_{\mathcal{B}^*(X)} : \mathcal{B}^*(X) \rightarrow \{0, 1\}.$$

Then  $\mu$  is a non-trivial Baire measure on  $X$ , and since we are assuming that  $X$  is realcompact, by Lemma 4.6 it has non-empty support. Let  $x \in \text{supp}(\mu)$  and let  $\{C_n : n \in \mathbb{N}\}$  be a local base at  $x$  in  $X$ . Without any loss of generality (since  $X$  is completely regular) we may assume that  $C_n \in \text{Coz}(X)$  for each  $n \in \mathbb{N}$  and  $C_1 \supseteq C_2 \supseteq \dots$ . Then

$$\mu(C_n) \rightarrow \mu\left(\bigcap_{n=1}^{\infty} C_n\right).$$

But this is a contradiction, as (since  $x \in \text{supp}(\mu)$ )  $\mu(C_n) = 1$  for each  $n \in \mathbb{N}$ , and since  $\mathcal{F}$  is free,

$$\bigcap_{n=1}^{\infty} C_n = \{x\} \notin \mathcal{F};$$

as otherwise,  $x \in F$  for each  $F \in \mathcal{F}$ , as  $\{x\} \cap F$  is non-empty, and thus (by our definition of  $\nu$ )

$$\mu\left(\bigcap_{n=1}^{\infty} C_n\right) = 0.$$

□

Obviously, every  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , where  $n \in \mathbb{N}$ , is realcompact. Thus, from Theorem 4.7 we obtain the following.

**Corollary 4.8.** *Let  $(\mathbb{R}^n, \mathcal{M})$  be the Lebesgue measurable space, where  $n \in \mathbb{N}$ . Then there is no free ultrafilter in  $\mathcal{M}$  with c.i.p.*

Recall that a cardinal  $\zeta$  is said to be *measurable* if there is a non-trivial  $\{0, 1\}$ -valued measure defined on the power set  $\mathcal{P}(X)$  of a set  $X$  of cardinality  $\zeta$  which vanishes at singletons. The following is well known.

**Theorem 4.9.** *In the measurable space  $(X, \mathcal{B})$ , let  $\mathcal{B} = \mathcal{P}(X)$ . Then the following are equivalent:*

- (1) *There is no free ultrafilter in  $\mathcal{B}$  with c.i.p.*
- (2)  *$X$  is of non-measurable cardinality.*

Our next theorem is sort of converse to Theorem 4.7. We need, however, some definitions and some lemmas first.

Let  $X$  be a completely regular topological space. For an open subset  $U$  of  $X$  the *extension of  $U$  to  $\beta X$*  is defined by

$$\text{Ex}_X U = \beta X \setminus \text{cl}_{\beta X}(X \setminus U).$$

The following lemma is well known (see Lemma 7.1.13 of [1] or Lemma 3.1 of [6]).

**Lemma 4.10.** *Let  $X$  be a completely regular topological space and let  $U$  and  $V$  be open subsets of  $X$ . Then*

- (1)  *$X \cap \text{Ex}_X U = U$ , and thus  $\text{cl}_{\beta X} \text{Ex}_X U = \text{cl}_{\beta X} U$ .*
- (2)  *$\text{Ex}_X(U \cap V) = \text{Ex}_X U \cap \text{Ex}_X V$ .*

The following lemma is proved by E. G. Skljarenko in [5]. It is rediscovered by E. K. van Douwen in [6].

**Lemma 4.11.** *Let  $X$  be a completely regular topological space and let  $U$  be an open subset of  $X$ . Then*

$$\text{bd}_{\beta X} \text{Ex}_X U = \text{cl}_{\beta X} \text{bd}_X U.$$

**Lemma 4.12.** *Let  $X$  be a completely regular topological space. If  $B$  is a subset of  $X$  with compact boundary then*

$$\text{cl}_{\beta X} B \setminus X = \text{Ex}_X(\text{int}_X B) \setminus X.$$

*Proof.* Since  $\text{bd}_X B$  is compact, we have

$$\text{cl}_{\beta X} \text{bd}_X B \subseteq \text{bd}_X B \subseteq X,$$

and therefore

$$\begin{aligned} \text{cl}_{\beta X} B \setminus X &= \text{cl}_{\beta X}(\text{int}_X B \cup \text{bd}_X B) \setminus X \\ &= (\text{cl}_{\beta X} \text{int}_X B \cup \text{cl}_{\beta X} \text{bd}_X B) \setminus X \\ &= (\text{cl}_{\beta X} \text{int}_X B \setminus X) \cup (\text{cl}_{\beta X} \text{bd}_X B \setminus X) = \text{cl}_{\beta X} \text{int}_X B \setminus X. \end{aligned}$$

By Lemma 4.10 we have

$$\begin{aligned} \text{cl}_{\beta X} \text{int}_X B \setminus X &= \text{cl}_{\beta X} \text{Ex}_X(\text{int}_X B) \setminus X \\ &= (\text{Ex}_X(\text{int}_X B) \cup \text{bd}_{\beta X} \text{Ex}_X(\text{int}_X B)) \setminus X \\ &= (\text{Ex}_X(\text{int}_X B) \setminus X) \cup (\text{bd}_{\beta X} \text{Ex}_X(\text{int}_X B) \setminus X). \end{aligned}$$

But by Lemma 4.11, we have

$$\text{bd}_{\beta X} \text{Ex}_X(\text{int}_X B) = \text{cl}_{\beta X} \text{bd}_X(\text{int}_X B).$$

On the other hand  $\text{bd}_X(\text{int}_X B) \subseteq \text{bd}_X B$ . Thus

$$\text{cl}_{\beta X} \text{bd}_X(\text{int}_X B) \subseteq \text{bd}_X B \subseteq X.$$

Combining these, we obtain the result.  $\square$

Note that if  $U$  is an open subset of a completely regular topological space  $X$ , then (since  $X$  is dense in  $\beta X$ ) we have

$$\text{cl}_{\beta X} U = \text{cl}_{\beta X}(U \cap X).$$

We use this simple observation in the following.

**Theorem 4.13.** *Let  $(X, \mathcal{O}, \mathcal{B})$  be a completely regular topological measurable space. Suppose that each  $B \in \mathcal{B}$  has compact boundary in  $X$ . If there is no free ultrafilter in  $\mathcal{B}$  with c.i.p. then  $X$  is realcompact.*

*Proof.* Suppose the contrary, i.e., suppose that  $X$  is not realcompact. Then  $X \neq \nu X$ . Let  $p \in \nu X \setminus X$ . Let

$$\mathcal{F} = \{B \in \mathcal{B} : p \in \text{cl}_{\beta X} B\}.$$

We show that  $\mathcal{F}$  is a free ultrafilter in  $\mathcal{B}$  with c.i.p., contradicting our assumption. Note that  $\emptyset \neq \mathcal{F} \subseteq \mathcal{B}$  and  $\emptyset \notin \mathcal{F}$ . Suppose that  $F, G \in \mathcal{F}$ . Then, by Lemmas 4.10 and 4.12 we have

$$\begin{aligned} p &\in (\text{cl}_{\beta X} F \cap \text{cl}_{\beta X} G) \setminus X \\ &= (\text{cl}_{\beta X} F \setminus X) \cap (\text{cl}_{\beta X} G \setminus X) \\ &= (\text{Ex}_X(\text{int}_X F) \setminus X) \cap (\text{Ex}_X(\text{int}_X G) \setminus X) \\ &= (\text{Ex}_X(\text{int}_X F) \cap \text{Ex}_X(\text{int}_X G)) \setminus X \\ &= \text{Ex}_X(\text{int}_X F \cap \text{int}_X G) \setminus X \\ &= \text{Ex}_X(\text{int}_X(F \cap G)) \setminus X = \text{cl}_{\beta X}(F \cap G) \setminus X. \end{aligned}$$

Therefore  $p \in \text{cl}_{\beta X}(F \cap G) \setminus X$  and thus  $F \cap G \in \mathcal{F}$ . Next, suppose that  $F \subseteq B$  for some  $B \in \mathcal{B}$  and  $F \in \mathcal{F}$ . Then

$$p \in \text{cl}_{\beta X} F \subseteq \text{cl}_{\beta X} B$$

and thus  $B \in \mathcal{F}$ . This shows that  $\mathcal{F}$  is a filter in  $\mathcal{B}$ . To show that  $\mathcal{F}$  is an ultrafilter, let  $B \in \mathcal{B}$  be such that  $B \cap F$  is non-empty for each  $F \in \mathcal{F}$ . If  $B \notin \mathcal{F}$  then  $p \notin \text{cl}_{\beta X} B$ . Thus  $p \in \text{cl}_{\beta X}(X \setminus B)$ , i.e.,  $X \setminus B \in \mathcal{F}$ . But this is not possible, as  $X \setminus B$  misses  $B$ . Therefore  $B \in \mathcal{F}$ . To show that  $\mathcal{F}$  is free, let  $x \in X$ . Then (since  $p \notin X$ ) there exist some disjoint open neighborhoods  $U$  and  $V$  of  $p$  and  $x$  in  $\beta X$ , respectively. Then

$$p \in \text{cl}_{\beta X} U = \text{cl}_{\beta X}(U \cap X)$$

and thus (since  $U \cap X \in \mathcal{O} \subseteq \mathcal{B}$ ) we have  $x \notin U \cap X \in \mathcal{F}$ . Therefore  $x \notin \bigcap \mathcal{F}$ . Thus  $\bigcap \mathcal{F} = \emptyset$  and  $\mathcal{F}$  is free. To show that  $\mathcal{F}$  has c.i.p., let  $F_1, F_2, \dots \in \mathcal{F}$ . Then  $p \in \text{cl}_{\beta X} F_n$  for each  $n \in \mathbb{N}$  and thus, since by Lemma 4.12 we have

$$\text{cl}_{\beta X} F_n \setminus X = \text{Ex}_X(\text{int}_X F_n) \setminus X$$

it follows that  $p \in \text{Ex}_X(\text{int}_X F_n)$ . For each  $n \in \mathbb{N}$ , let  $f_n : \beta X \rightarrow [0, 1]$  be continuous and such that

$$f_n(p) = 0 \text{ and } f_n|(\beta X \setminus \text{Ex}_X(\text{int}_X F_n)) \equiv 1.$$

Then

$$p \in Z = \bigcap_{n=1}^{\infty} Z(f_n) \in \mathcal{Z}(\beta X).$$

If  $Z \cap X = \emptyset$  then  $Z \subseteq \beta X \setminus vX$ , which is a contradiction, as  $p \in vX$ . Thus, using Lemma 4.10 we have

$$\begin{aligned} \emptyset \neq Z \cap X &= \bigcap_{n=1}^{\infty} Z(f_n) \cap X \\ &\subseteq \bigcap_{n=1}^{\infty} \text{Ex}_X(\text{int}_X F_n) \cap X = \bigcap_{n=1}^{\infty} \text{int}_X F_n \subseteq \bigcap_{n=1}^{\infty} F_n. \end{aligned}$$

Therefore

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

This show that  $\mathcal{F}$  has c.i.p. □

*Remark 4.14.* In the case when  $X$  is normal, using Lemma 4.6, one can give an alternative proof for Theorem 4.13. To show this assume that there exists a non-trivial  $\{0, 1\}$ -valued Baire measure  $\nu$  on  $X$  whose support is empty. Let

$$\mathcal{F} = \{B \in \mathcal{B} : \nu^*(X \setminus B) = 0\}$$

in which  $\nu^*$  is the outer measure induced by  $\nu$ . We verify that  $\mathcal{F}$  is a free ultrafilter in  $\mathcal{B}$  with c.i.p. Obviously,  $\mathcal{F}$  is non-empty (as  $X \in \mathcal{F}$ ) and  $\emptyset \notin \mathcal{F}$  (as  $\nu$  is non-trivial). Note that for any  $F, G \in \mathcal{F}$  since

$$\nu^*(X \setminus (F \cap G)) = \nu^*((X \setminus F) \cup (X \setminus G)) \leq \nu^*(X \setminus F) + \nu^*(X \setminus G) = 0$$

we have  $F \cap G \in \mathcal{F}$ . Also, if  $F \subseteq B$  for some  $F \in \mathcal{F}$  and  $B \in \mathcal{B}$  (since  $X \setminus B \subseteq X \setminus F$ ) we have

$$\nu^*(X \setminus B) \leq \nu^*(X \setminus F) = 0$$

and thus  $B \in \mathcal{F}$ . Therefore  $\mathcal{F}$  is a filter in  $\mathcal{B}$ . To show that  $\mathcal{F}$  is an ultrafilter, let  $B \in \mathcal{B}$  be such that  $B \cap F$  is non-empty for each  $F \in \mathcal{F}$ . Since  $\text{supp}(\nu) = \emptyset$ , for each  $x \in X$  there exists some  $U_x \in \text{Coz}(X)$  such that  $x \in U_x$  and  $\nu(U_x) = 0$ . By compactness of  $\text{bd}_X B$  there exist  $x_1, \dots, x_n \in X$  such that

$$\text{bd}_X B \subseteq \bigcup_{i=1}^n U_{x_i}.$$

Let

$$U = \bigcup_{i=1}^n U_{x_i} \in \text{Coz}(X).$$

Then

$$\text{cl}_X B \setminus U \subseteq \text{cl}_X B \setminus \text{bd}_X B \subseteq \text{int}_X B$$

and thus (since we are assuming that  $X$  is normal) by the Urysohn Lemma there exists a continuous  $f : X \rightarrow [0, 1]$  such that

$$f|(X \setminus \text{int}_X B) \equiv 0 \text{ and } f|(\text{cl}_X B \setminus U) \equiv 1.$$

Now, if  $B \notin \mathcal{F}$  (by the definition of  $\mathcal{F}$ ) we have  $\nu^*(X \setminus B) = 1$ . Thus, since  $X \setminus B \subseteq Z(f)$  we have

$$\nu(Z(f)) = \nu^*(Z(f)) = 1$$

and therefore

$$\nu(\text{Coz}(f)) = 1 - \nu(Z(f)) = 0.$$

Now, since

$$B \subseteq \text{cl}_X B \subseteq (\text{cl}_X B \setminus U) \cup U \subseteq \text{Coz}(f) \cup U = \text{Coz}(f) \cup \bigcup_{i=1}^n U_{x_i}$$

we have

$$\nu^*(X \setminus (X \setminus B)) = \nu^*(B) \leq \nu(\text{Coz}(f)) + \sum_{i=1}^n \nu(U_{x_i}) = 0.$$

Therefore (by the definition of  $\mathcal{F}$ )  $X \setminus B \in \mathcal{F}$ , which is not possible, as it misses  $B$ . This contradiction shows that  $\mathcal{F}$  is an ultrafilter. It remains to show that  $\mathcal{F}$  has c.i.p. But this follows easily, as if  $F_1, F_2, \dots \in \mathcal{F}$ , then since

$$\nu^*\left(X \setminus \bigcap_{i=1}^{\infty} F_i\right) = \nu^*\left(\bigcup_{i=1}^{\infty} (X \setminus F_i)\right) \leq \sum_{i=1}^{\infty} \nu^*(X \setminus F_i) = 0$$

we have

$$\bigcap_{i=1}^{\infty} F_i \in \mathcal{F}.$$

Finally, note that  $\mathcal{F}$  is free, as for each  $x \in X$  since  $\nu(U_x) = 0$  (with  $U_x$  as defined in the above) we have  $X \setminus U_x \in \mathcal{F}$ , and thus  $x \notin \bigcap \mathcal{F}$ .

**Theorem 4.15.** *Let  $(Y, \mathcal{U}, \mathcal{C})$  be a first-countable Hausdorff topological measurable space. Let  $(X, \mathcal{B}) \subseteq (Y, \mathcal{C})$  and let  $Y \setminus X$  be Lindelöf. If there is no free ultrafilter in  $\mathcal{B}$  with c.i.p. then there is no free ultrafilter in  $\mathcal{C}$  with c.i.p.*

*Proof.* For each  $y \in Y$ , let

$$\{V_n^y : n \in \mathbb{N}\}$$

be an open base at  $y$  in  $Y$ . Suppose to the contrary that there exists a free ultrafilter  $\mathcal{H}$  in  $\mathcal{C}$  with c.i.p.

**Claim 1.** *For each  $y \in Y$  there exist some  $n_y \in \mathbb{N}$  and  $H_y \in \mathcal{H}$  such that  $V_{n_y}^y \cap H_y = \emptyset$ .*

*Proof of Claim 1.* Suppose the contrary, i.e., suppose that for some  $y \in Y$  the set  $V_n^y \cap H$  is non-empty for each  $n \in \mathbb{N}$  and  $H \in \mathcal{H}$ . Note that since  $\mathcal{H}$  is an ultrafilter in  $\mathcal{C}$  this implies that  $V_n^y \in \mathcal{H}$  for each  $n \in \mathbb{N}$ . Since  $Y$  is Hausdorff, we have

$$\bigcap_{n=1}^{\infty} V_n^y = \{y\},$$

and since  $\mathcal{H}$  has c.i.p., we have

$$H \cap \{y\} = H \cap \bigcap_{n=1}^{\infty} V_n^y \neq \emptyset$$

i.e.,  $y \in H$  for each  $H \in \mathcal{H}$ , contradicting the fact that  $\mathcal{H}$  is free. This shows Claim 1.

**Claim 2.**  *$H \cap X$  is non-empty for each  $H \in \mathcal{H}$ .*

*Proof of Claim 2.* Suppose the contrary, i.e., suppose that  $H \subseteq Y \setminus X$  for some  $H \in \mathcal{H}$ . Since

$$Y \setminus X \subseteq \bigcup \{V_{n_y}^y : y \in Y \setminus X\}$$

and  $Y \setminus X$  is Lindelöf, we have

$$Y \setminus X \subseteq \bigcup_{i=1}^{\infty} V_{n_{y_i}}^{y_i}$$

for some  $y_1, y_2, \dots \in Y \setminus X$ . Now, by Claim 1 we have

$$\begin{aligned} H \cap \bigcap_{j=1}^{\infty} H_{y_j} &\subseteq \left( \bigcup_{i=1}^{\infty} V_{n_{y_i}}^{y_i} \right) \cap \bigcap_{j=1}^{\infty} H_{y_j} \\ &= \bigcup_{i=1}^{\infty} \left( V_{n_{y_i}}^{y_i} \cap \bigcap_{j=1}^{\infty} H_{y_j} \right) \subseteq \bigcup_{i=1}^{\infty} (V_{n_{y_i}}^{y_i} \cap H_{y_i}) = \emptyset \end{aligned}$$

contrary to the fact that  $\mathcal{H}$  has c.i.p.

Let

$$\mathcal{A} = \{H \cap X : H \in \mathcal{H}\}.$$

Then  $\mathcal{A} \subseteq \mathcal{B}$  (as  $(X, \mathcal{B}) \subseteq (Y, \mathcal{C})$  and  $\mathcal{H} \subseteq \mathcal{C}$ ) and by Claim 2 we have  $\emptyset \notin \mathcal{A}$ . Since  $\mathcal{A}$  is obviously closed under finite intersections, as  $\mathcal{H}$  is so,  $\mathcal{A}$  is a filter-base in  $\mathcal{B}$ . Let  $\mathcal{F}$  be an ultrafilter in  $\mathcal{B}$  such that  $\mathcal{A} \subseteq \mathcal{F}$ . Since  $\mathcal{H}$  is free, we have

$$\bigcap \mathcal{F} \subseteq \bigcap \mathcal{A} = \bigcap \mathcal{H} \cap X = \emptyset$$

i.e.,  $\mathcal{F}$  also is free. To show that  $\mathcal{F}$  has c.i.p., let  $F_1, F_2, \dots \in \mathcal{F}$ . Let  $F_n = C_n \cap X$  where  $C_n \in \mathcal{C}$  for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$  and  $H \in \mathcal{H}$ , since  $H \cap X \in \mathcal{A} \subseteq \mathcal{F}$ , we have

$$\emptyset \neq H \cap X \cap F_n = H \cap X \cap C_n \subseteq H \cap C_n$$

and thus, since  $\mathcal{H}$  is an ultrafilter in  $\mathcal{C}$ , we have  $C_n \in \mathcal{H}$ . Now, for each  $H \in \mathcal{H}$ , since  $\mathcal{H}$  has c.i.p., we have

$$H \cap \bigcap_{n=1}^{\infty} C_n \neq \emptyset,$$

and therefore

$$\bigcap_{n=1}^{\infty} C_n \in \mathcal{H}.$$

By Claim 2 we have

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} C_n \cap X \neq \emptyset.$$

This shows that  $\mathcal{F}$  is a free ultrafilter in  $\mathcal{B}$  with c.i.p., which is a contradiction.  $\square$

Note that in the above proof we only need  $Y$  to be first-countable at the points of  $Y \setminus X$ .

5. EXAMPLES OF MEASURE SPACES  $(X, \mathcal{B}, \mu)$  WITH AN ARBITRARILY LARGE NUMBER OF FREE ULTRAFILTERS IN  $\mathcal{B}$  HAVING C.I.P.

**Example 5.1.** *Let  $\zeta$  be a cardinal. Then there exists a measure space  $(Z, \mathcal{D}, \nu)$  having at least  $\zeta$  free ultrafilters in  $\mathcal{D}$  with c.i.p.*

*Proof.* Let  $(X, \mathcal{B}, \mu)$  be a measure space in which  $\mu$  is a non-trivial  $\{0, 1\}$ -valued measure which (is defined and) vanishes at singletons. Let  $(Y, \mathcal{C}, \lambda)$  be a  $\sigma$ -finite measure space in which singletons are measurable and such that  $\text{card}(Y) \geq \zeta$ . By Proposition 4.1 there exists a free ultrafilter  $\mathcal{F}$  in  $\mathcal{B}$  with c.i.p. To see this, simply let

$$\mathcal{A} = \{\{x\} : x \in X\}$$

and observe that (since  $\mu$  is non-trivial)

$$\mu\left(\bigcup \mathcal{A}\right) = \mu(X) = 1 \neq 0 = \sup_{x \in X} \mu(x) = \sup_{A \in \mathcal{A}} \mu(A).$$

For each  $y \in Y$ , let  $\mathcal{H}_y$  be an ultrafilter in  $\mathcal{B} \times \mathcal{C}$  such that

$$\{F \times \{y\} : F \in \mathcal{F}\} \subseteq \mathcal{H}_y.$$

By the proof of Theorem 4.5 the ultrafilter  $\mathcal{H}_y$ , for each  $y \in Y$ , is free and has c.i.p. Note that  $\mathcal{H}_y$ 's are distinct if  $y \in Y$  are distinct. The measure space  $(X \times Y, \mathcal{B} \times \mathcal{C}, \mu \times \lambda)$  has the desired property.  $\square$

## 6. QUESTIONS

We conclude this article with the following questions.

**Question 6.1.** *In Theorem 4.7, does the converse hold? More precisely, for a first-countable topological measurable space  $(X, \mathcal{O}, \mathcal{B})$  does the non-existence of any free ultrafilter in  $\mathcal{B}$  with c.i.p. imply its realcompactness?*

**Question 6.2.** *It is known that each finite measure space can be embedded in a perfect measure space. Even more, each finite measure space can be embedded in a compact (in the sense of Marczewski) measure space. Find the corresponding choices of  $\mathbb{U}$ ,  $S_{\mathcal{U}}$  where  $\mathcal{U} \in \mathbb{U}$ ,  $(Z, \mathcal{D})$  and  $\{\mathcal{D}_B : B \in \mathcal{B}\}$  in Theorem 2.2 for any such embeddings.*

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## REFERENCES

- [1] R. Engelking, *General Topology*, Second edition, Heldermann Verlag, Berlin, 1989.
- [2] R.G. Gardner and W.F. Pfeffer, Borel measures, in: K. Kunen and J.E. Vaughan (Eds.), *Handbook of Set-theoretic Topology*, Elsevier, Amsterdam, 1984, pp. 961–1043.
- [3] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer-Verlag, New York-Heidelberg, 1976.
- [4] P.R. Halmos, *Measure Theory*, D. Van Nostrand Company, Inc., New York, 1950.
- [5] E.G. Skljarenko, Some questions in the theory of bicompatifications, (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **26** (1962), 427–452.
- [6] E.K. van Douwen, Remote points, *Dissertationes Math. (Rozprawy Mat.)* **188** (1981), 45 pp.

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